A chaos and fractal theory based nonlinear dynamical model for China stock market

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Abstract: The China stock market is proved to be a chaotic system, a nonlinear dynamical model is established based on the study of the nonlinear dynamical properties of Shanghai stock, composite index sequence by using chaos and fractal theory. The phase space of the stock sequence is reconstructed and the correlation dimension is analyzed, which indicates that the dynamical system has finite degree of freedom. The nonlinear evolution mechanism is observed and the initial value sensitive characteristic of the system is demonstrated through Lyapunov exponent analysis. Finally, the stock sequence is reconstructed by using finite degree of freedom based fractal interpolation algorithm and gaining reasonably accurate replications. The experimental results indicate that the nonlinear dynamical model is more effective to describe the China stock market than the conventional "random walk" theory based stochastic models.

Key words: chaos and fractal theory; stock market; finite degree of freedom; Lyapunov exponent; fractal interpolation

1. Introduction

Most of the conventional financial models are based on the famous "random walk" theory, which was proposed by French mathematician Louis Bachelier in his Ph.D. dissertation "The Theory of Speculation" in 1900. His theory indicated that the price is normally distributed, the changes of the price are independent in statistical and their distribution is a normal bell curve. According to his theory, the price should fluctuate slightly in a small range, the probability of large shakeout should seldom happen, but the frequently happened violent changes in modern stock market indicates that current securities theory cannot give a correct explanation.

In 1999, Professor B.B.Mandelbrot, the academician of American Academy and founder of fractal theory, published revolutionary paper "A multi-fractal walk down Wall Street" in "Scientific American" [1]. He proposed a new multi fractal model to describe the fluctuation of the securities market. His theory indicates that the price fluctuation accords with "multi-fractal walk" principle, and a general model was constructed to simulate the stock market trading process with huge fluctuation. It accurately describes the relationship of the up and down fluctuation, providing a probability estimation of the securities market trend, revealing the essential of the frequent fluctuation of securities market, providing a deterministic analysis method for the study of unpredictable financial system.

Chaos and fractal has a tight relationship, the orbit or attractor of a chaotic dynamical system is a fractal set [2-4]. A large number of self-similar fractal models are proposed recently but few scholars set up financial models combining with chaos theory. This paper proves China stock market is a chaotic system and proposes a finite degree of freedom based fractal interpolation model. The Shanghai stock composite index is reconstructed gaining reasonably accurate replications, which resolve the disadvantage of conventional "random-walk" model essentially.

2. Phase space reconstruction

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The complex stochastic behavior of a chaotic system is determined by the nonlinear equations of a determined system that has finite independent free variables and the phase space is a finite dimension status space constructed by these independent variables. The track of status points represents the dynamical evolvement process of a system and the system will shrink to an attractor with low dimension in the end. In general, the observation of a complex system is by sampling a certain variable of the system in a period of time and the achieved time sequence contains all the system variables concerned with the dynamical process. The phase space can be reconstructed by embedding the sampled scalar quantity sequence into the system status space. The embedding process is an equipollence and reversible transform, although it may cause the orbit of the dynamical system in phase space distortion, the system properties is reserved, the topology is not changed and the order of status point is kept. So the characteristic of the original dynamical system can be achieved from the embedded phase space.

This paper uses time delay algorithm that proposed by Takens to reconstruct the phase space[5]. Let $\{y(n), n = 1, 2, \ldots, N\}$ be sampled time sequence, embedding it to m dimension Euclid space R^m and achieve a points collection $J(m)$. The elements in $J(m)$ is marked as

$$Y_{m,\tau} = \{y(n), y(n+\tau)......y(n+(m-1)\tau)\},\ n = 1,2,...,N_m,$$

in which τ is delay and $N_m = n - (m - 1)\tau$ is the total amount of reconstructed status points. Taken's embedded theory indicates that even the number and characteristics of the independent variables in original system are difficult to determine, if the embedded dimension m is large enough, the reconstructed phase space is isomorphic with the original space. The dimension, Lyapunov exponent and even the nonlinear function of the original system can be calculated. The time delay τ must be selected properly when reconstructing the phase space. If τ is too large, the relationship of neighbor points in status space is lost, which will lead the reconstructed space disorder and ruleless. If τ is too small, the two neighbor points are too closed to be an independent coordinate. This paper uses auto correlation function to determine delay τ. The auto correlation function of time sequence x_t is defined as

$$r_\tau = \frac{\sum_{t=\tau+1}^{n}(x_t - \bar{x})(x_{t-\tau} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \tag{1}$$

in which $\bar{x}$ is the mean of sequence and r_τ is the auto correlation coefficient of time delay τ. The tendency chart of Shanghai stock composite index from 1-1th-1993 to 12-31th-2003 is shown in Fig.1 and the corresponding auto correlation function diagram is shown in Fig. 2.

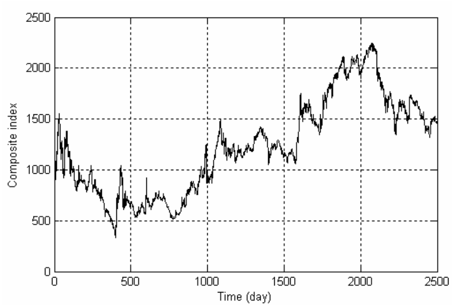

Fig. 1 Tendency chart of Shanghai stock composite index

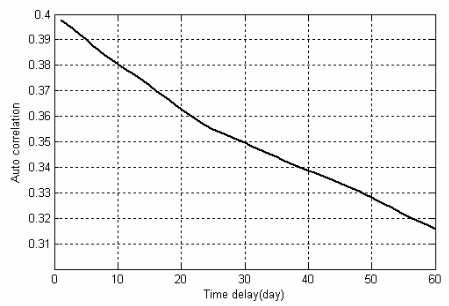

Fig. 2 Auto correlation function of Shanghai stock composite index

Time delay τ should be set to the value that the corresponding auto correlation coefficient first smaller than $1/e \approx 0.368$, so the optimized time delay τ should be set to 17 according to Fig. 2.

3. Finite degree of freedom analysis

A pure stochastic system has infinite degree of freedom and behaves pure "randomness" that could not be predicted. While different from the pure stochastic system, the stochastic property of a chaotic system is due to the nonlinear properties it has. A chaotic dynamical system only has finite independent free variables and can be described by nonlinear equation systems, which indicate the deterministic property of a chaotic system. The correlation dimension of the attractor of a chaotic system can be achieved by analysis of the change tendency of correlation dimension $D(m)$ with the increase of embedded dimension. The correlation dimension of a stochastic system will increase along with the increase of embedded dimension m due to the infinite degree of freedom it has. While for a chaotic system, the correlation dimension is limited, which can be used as a proof that a chaotic system has finite degree of freedom.

The correlation dimension of Shanghai stock index sequence is calculated by using G-P algorithm based on the reconstructed phase space [6]. Let r_{ij} be the Euclid distance of any two status points X_i, X_j in $J(m)$, then the number of the points that the distance of two points less than radius r is

$$C(r) = \frac{1}{N_m^{\ 2}} \sum_{i=1}^{N_m} \sum_{j=1}^{N_m} H(r - r_{ij}) \qquad (2)$$

in which H is Heaviside function:

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}. \qquad (3)$$

Thus the correlation dimension of m dimension embedded space is

$$D(m) = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}. \qquad (4)$$

In order to calculate $D(m)$, the double logarithmic curve of $\ln C(r)$ and $\ln r$ is drawn. The curve will have a beeline portion corresponding to a proper range of r, and the slope of the line is the correlation dimension $D(m)$. For a chaotic dynamical system with finite dimension attractor, the independent free variable is limited, so $D\ (m)$ will converge to a constant with the increase of m, namely

$$D = \lim_{m \to \infty} D(m) \qquad (5)$$

in which D is the correlation dimension of the system attractor.

The fitting curve of system embedded dimension with correlation dimension of Shanghai stock composite index is shown in Fig. 3. From Fig. 3 we can get that the optimized embedded dimension is 15 and the corresponding correlation dimension is 1.61. Experiment results indicate that although the stock sequence is stochastic and behaves very complexly, the dimension of the phase space is bounded and the phase space is controlled by a nonlinear dynamical system with limited dimension. There exists deterministic properties in stock system, which is different from conventional stochastic model that the system has infinite degree of freedom.

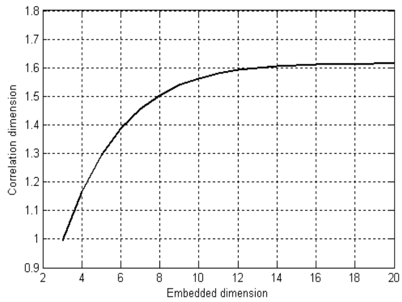

Fig. 3 Curve of system embedded dimension with correlation dimension

4. Lyapunov exponent analysis

The time-space evolvement properties can be further studied based on the reconstructed phase space. For a deterministic linear system, the signal at certain time is steady and predictable, the status evolvement orbit is smooth and differentiable. While different from deterministic linear system, the behavior of a chaos system is very complex and long term unpredictable due to its initial value sensitive property. To evaluate the departure speed of neighbor points in phase space, Lyapunov exponent is introduced,

$$\sigma = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i} \tag{6}$$

in which σ presents the average departure exponent of each iteration. When $\sigma < 0$, the neighbor points will unite to one point, this corresponds to immovable points or periodic movement. While $\sigma > 0$ indicates the two points will depart exponentially, this corresponds to chaotic process. Thus only if the attractor has one positive Lyapunov exponent, the dynamical system is chaotic. The maximum Lyapunov exponent calculation algorithm is as follows[7]:

(I) In extended m dimension phase space, select initial phase point $A(t_0)$ as reference point, finding point $B(t_0)$ which has minimum Euclid distance to $A(t_0)$ in other phase points, and the distance is marked as $L(t_0)$. Assume that at $t_1 = t_0 + \Delta t$, $A(t_0)$ point evolves to $A(t_1)$ and $B(t_0)$ to $B(t_1)$, the distance between them is $\overline{A(t_1)B(t_1)} = l(t_1)$. Let λ_1 presents the departure exponent in Δt interval, thus

$$l(t_1) = L(t_0)2^{\lambda_1 \Delta t} \tag{7}$$

namely,

$$\lambda_1 = \frac{1}{t_1 - t_0}\log_2\frac{l(t_1)}{L(t_0)} = \frac{1}{\Delta t}\log_2\frac{l(t_1)}{L(t_0)} \tag{8}$$

(II) Find a point $B(t_1')$ which is one of the enough closer neighbor points to $A(t_1)$ and satisfies $\theta_1 = (B(t_1) - A(t_1),\hat{} B(t_1') - A(t_1))$ to substitute $B(t_1)$, let the distance $\overline{A(t_1)B(t_1')} = L(t_1)$. If $B(t_1')$ could not be found, then let $B(t_1') = B(t_1)$ and $L(t_1) = l(t_1)$. Assume at $t_2 = t_1 + \Delta t$, $A(t_1)$ point evolves to $A(t_2)$ and $B(t_1')$ to $B(t_2)$, the distance $\overline{A(t_1)B(t_2)} = l(t_2)$, thus

$$\lambda_2 = \frac{1}{t_2 - t_1}\log_2\frac{l(t_2)}{L(t_1)} = \frac{1}{\Delta t}\log_2\frac{l(t_2)}{L(t_1)} \tag{9}$$

(III) Repeating evolvement process (II) until the orbit of the search points covers the whole set $J(m)$, and then use the average of the exponent increase ratio λ_n as the maximum Lyapunov exponent of m dimension phase space, namely

$$\lambda_{\max}(m) = \frac{1}{N\Delta t}\sum_{i=1}^{N}\log_2\frac{l(t_i)}{L(t_{i-1})} = \frac{1}{N(t_i - t_{i-1})}\sum_{i=1}^{N}\log_2\frac{L(t_i)}{L(t_{i-1})} \tag{10}$$

in which Δt is step length and N is total amounts of steps.

(IV) Increase the embedded dimension m, repeating step (I) ~ (III), until $\lambda_{\max}(m)$ reaches a steady value, let

$$\lambda_{\max} = \lim_{m\to\infty}\lambda_{\max}(m) \tag{11}$$

in which $\lambda_{\max}$ is the maximum Lyapunov exponent of the dynamical system.

Based on above algorithm, the calculated maximum Lyapunov exponent of Shanghai stock composite index is $\lambda_{\max} = 0.133$, which indicates that although the system has finite degree of freedom, the time sequence changes dramatically and the system is almost unpredictable, which shows the complex stochastic properties in a deterministic system.

5. The reconstruction of stock index sequence based on fractal interpolation

Interpolation is a common means to restore the microcosmic details of a large scale measured data. The microcosmic details of stock index sequence is nonlinear and hard to describe by using conventional method, while fractal theory is just very suitable to solve this kind of question, thus the sequence can be reconstructed effectively by using fractal interpolation method. This paper uses FBM (Fractional Brownian Motion) based fractal interpolation algorithm[8], which will be described in the following.

Let $B_H(t)$ be a time sequence, if $B_H(t_2) - B_H(t_1)$ satisfies 0 mean Gaussian distribution, then $B_H(t)$ is FBM and the variance satisfies:

$$\mathrm{var}[B_H(t_2) - B_H(t_1)] = \sigma^2|t_2 - t_1|^{2H} \tag{12}$$

H is Hurst exponent, which has the following relationship with system fractal dimension D

$$D = 2 - H \tag{13}$$

FBM has statistical self-similar property, the stochastic midpoint displacement algorithm is used to implement the interpolation and the algorithm is a simple iteration process. For a FBM time sequence $B_H(t)$, if $B_H(0)$ and $B_H(1)$ are known, the value of midpoint $B_H(1/2)$ can be calculated by Eq.14.

$$B_H(1/2) = [B_H(1) + B_H(0)]/2 + \Delta_1 \tag{14}$$

in which Δ_1 is a Gauss variable with mean 0 and variance $\delta_1{}^2$.

$$\begin{aligned}\delta_1^2 &= \mathrm{var}[B_H(1/2) - B_H(0)] - \frac{1}{4}\mathrm{Var}[B_H(1) - B_H(0)] \\ &= \frac{\sigma^2}{2^{2H}}(1 - 2^{2H-2})\end{aligned} \tag{15}$$

Similarly, $B_H(1/4)$ can be calculated by using $B_H(0)$ and $B_H(1/2)$ as two new ends and $B_H(3/4)$ by using $B_H(1/2)$ and $B_H(1)$ in the second level iteration. The rest can be deduced by analog, in n level iteration,

$$\sigma_n^2 = \frac{\sigma^2}{(2^n)^{2H}}(1 - 2^{2H-2}) \tag{16}$$

The reconstructed stock index sequence with 1000 points is shown in Fig. 4. From Fig.4 we can see, the reconstructed sequence has very fine and realistic structure due to the self-similarity property that fractal geometry has, and the signal is precisely reconstructed, which further proves that China stock market has fractal property.

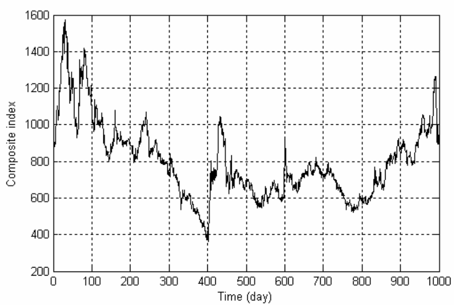

Fig. 4 Reconstructed stock index sequence with 1000 points

6. Conclusions

This paper provides a new approach to study the space-time evolution rules of stock market by using fractal and chaos theory. Compared with conventional "random walk" theory based on stochastic models, the nonlinear model proposed effectively reveals the essential rules of China stock market, and the stock index sequence can be reconstructed precisely by using finite degree of freedom based on fractal interpolation, which further proves the fractal property that the stock sequence has.

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